Reconstruction of discrete sets from two or more X-rays in any direction

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Reconstruction of Discrete Sets from Two or More Projections in Any Direction

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Abstract. During the workshop entitled "Discrete Tomography", held in Volkrange on March 22, 1999, A. Kuba presented the open problem of reconstructing discrete sets satisfying the properties of connectivity and convexity by projections taken along many directions. In this paper, we study this problem, considering a similar property of discrete sets: the Q-convexity. In fact this property contains a certain kind of connectivity and convexity. The main result of this paper is a polynomial-time algorithm which is able to reconstruct Q-convex sets from their projections, when the directions of the projections and the ones of the Q-convexity are the same. Moreover, the algorithm works for any finite number of directions.
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1 Introduction

A discrete set is a not-empty finite subset of the integer lattice $\mathbb{Z}^2$. The *projection* of a discrete set F in a direction p is the function $rp : \mathbb{Z} \to \mathbb{N}$ giving the number of F points on each line parallel to this direction, defined by: $rp(i) = |\{N \in F \,|\, p(N) = i\}|$. Many authors have studied the case of determining a discrete set from its projections in the horizontal and vertical directions and, in particular, there are polynomial algorithms to reconstruct special sets having some convexity and/or connectivity properties like, for example, horizontally and vertically convex polyominoes [2, 3, 4]. In this paper we study the inverse problem of reconstructing discrete sets from their projections given in any pair of rational directions (p, q), defined by: $p(M) = ax_M + by_M$ and $q(M) = cx_M + dy_M$, with $a, b, c, d \in \mathbb{Z}$. Without loss of generality we assume that $ad - bc \neq 0$, $gcd(a, b) = 1$, $gcd(c, d) = 1$. We present a new class of subsets of $\mathbb{Z}^2$and we provide an algorithm solving the problem for that class in polynomial time. We recall that the general problem of reconstructing finite subsets of n-dimensional sets that are only accessible via their projections in a finite set $\mathcal{D}$ of three or more directions is NP-complete as shown in [8]. In [1] the authors reconstruct connected discrete sets which are convex in the directions of the projections and they assume that $(1, 0)$ and $(0, 1)$ are in $\mathcal{D}$. We point out that we do not impose any constraints on $\mathcal{D}$. Moreover, our algorithm can be extended in order

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to work with more projections. Finally, since the definition of the new class is given in terms of $\mathcal{D}$, the problems studied in [2, 3, 4] are solvable as special case of our problem.

2 Definitions and notations

Let $\mathcal{D} = \{p, q\}$ be a set of two prescribed rational directions. Furthermore we call *p-line* and *q-line* the lines having equation $p(M) = const$ and $q(M) = const$ for each $M \in \mathbb{Z}^2$, respectively. We point out that if $\delta = |\det(p, q)| = |ad - bc| \neq 1$, the intersection of a p-line and a q-line is not always in $\mathbb{Z}^2$. In [7] the authors give a condition to determine whether the intersection of these lines is a point of $\mathbb{Z}^2$.
A point M belongs to $\mathbb{Z}^2$ if and only if $j \equiv ki \pmod{\delta}$, where $p(M) = i,\ q(M) = j$ and $k = (cu + dv)\text{sign}(ad - bc) \pmod{\delta}$, $au + bv = 1$. Thus, let us consider the point M; it selects the following four zones (see Fig. 1):

$$
\begin{aligned}
Z_0(M) &= \{N \in \mathbb{Z}^2 : \ p(N) \leq p(M) \text{ and } q(N) \leq q(M)\},\\
Z_1(M) &= \{N \in \mathbb{Z}^2 : \ p(N) \geq p(M) \text{ and } q(N) \leq q(M)\},\\
Z_2(M) &= \{N \in \mathbb{Z}^2 : \ p(N) \geq p(M) \text{ and } q(N) \geq q(M)\},\\
Z_3(M) &= \{N \in \mathbb{Z}^2 : \ p(N) \leq p(M) \text{ and } q(N) \geq q(M)\}.
\end{aligned}
$$

Let F be a subset of $\mathbb{Z}^2$.

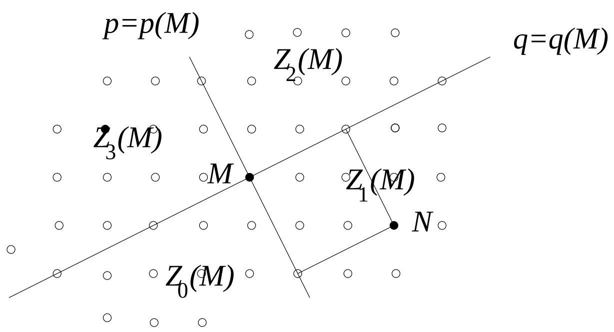

Figure 1: The integer lattice $\mathbb{Z}^2$ and the four zones defined by the point M, when $p = 2x + y$ and $q = -x + 2y$.

Definition 2.1 *F is Q-convex around $\{p, q\}$ if $Z_t(M) \cap F \neq \emptyset$ for all $t \in [0, 3]$ implies $M \in F$.*

We denote the class of discrete sets which are Q-convex around $\mathcal{D}$ by $\mathcal{F}$. The main interest of this definition is that it carries out both convexity and connectivity properties. More in detail, in [5] the author shows that a subset of $\mathbb{Z}^2$ is in $\mathcal{F}$ if it is the intersection between $\mathbb{Z}^2$ and a subset of $\mathbb{R}^2$ which is connected (according to the definition in the continuous case) and *simply convex* along all the directions of $\mathcal{D}$. We recall that a subset $\mathcal{E}$ of $\mathbb{R}^2$ is simply convex along p if, for each pair of points (M, N) of $\mathcal{E}$ such that $p(M) = p(N)$, the segment $[MN] \subset \mathcal{E}$. We can give a similar definition on $\mathbb{Z}^2$. Let $\mathcal{E}$ be a subset of $\mathbb{Z}^2$; $\mathcal{E}$ is simply convex along p if for each pair of points (M, N) of $\mathcal{E}$ such that $p(M) = p(N)$, the segment $[MN] \cap \mathbb{Z}^2 \subset \mathcal{E}$. Fig. 2 shows some examples of discrete sets having different kinds of convexity, when the considered directions are $p = x$ and $q = y$.

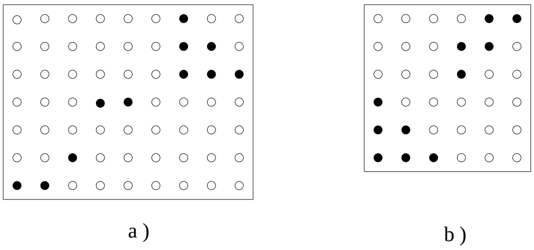

Figure 2: a) A discrete set Q-convex around $(1, 0)$ and $(0, 1)$. b) A discrete set simply convex along $(1, 0)$ and $(0, 1)$, but not Q-convex.

Definition 2.2 *F is* indivisible *for the direction p, or p-*indivisible*, if $\{i \in \mathbb{Z} : rp(i) > 0\}$ is made up of consecutive integers.*

By the definition, a p-indivisible discrete set has at least a point in each line $p = i$ parallel to the direction p. If F is p and q-indivisible with $\mathcal{D} = \{p, q\}$, we say that F is *$\mathcal{D}$-indivisible* or shortly, *indivisible*. The discrete set shown in Fig. 2a) is not indivisible, whereas that in Fig. 2b) is indivisible. An example of an indivisible discrete set which is simply convex in the directions $p = x - y$ and $q = x + y$, but not Q-convex, is given in Fig. 3. In case $p = (1, 0)$ and $q = (0, 1)$, F belongs to $\mathcal{F}$ and is indivisible if and only if

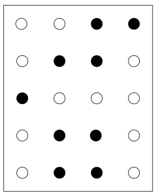

Figure 3: An indivisible discrete set simply convex along $(1, 1)$ and $(-1, 1)$, but not Q-convex.

F is simply convex in directions p and q and 8-connected (see [5]).

Let $pmin = \min\{p(N) : N \in F\}$ and $pmax = \max\{p(N) : N \in F\}$ and similarly, $qmin = \min\{q(N) : N \in F\}$ and $qmax = \max\{q(N) : N \in F\}$. In this paper we assume $pmax - pmin + 1 = m$ and $qmax - qmin + 1 = n$. The discrete set to be reconstructed is contained in the discrete parallelogram (see Fig. 4):

$$\Delta = \{N \in \mathbb{Z}^2 : pmin \leq p(N) \leq pmax, \ qmin \leq q(N) \leq qmax\}.$$

We call *bases* of F the points of the boundary of Δ belonging to F. Formally, let $j_0 = \min\{j : (pmin, j) \in F\}$ and $j_1 = \max\{j : (pmin, j) \in F\}$, and moreover let $j'_0 = \min\{j : (pmax, j) \in F\}$ and $j'_1 = \max\{j : (pmax, j) \in F\}$. We call *minimum p-base* and *maximum p-base* of F the sets of points:

$$PMIN = \{N = (pmin, j) : j_0 \le j \le j_1\}$$

and

$$PMAX = \{N = (pmax, j) : j'_0 \le j \le j'_1\},$$

respectively. The difference between j_1 and j_0 plus one is the size of the minimum p-base. The positions of the p-bases of F are given in terms of the indices $j_0,\ j_1,\ j'_0,\ j'_1$. So, we can have $j_0 \le j'_1$ or $j'_0 \le j_1$. The corresponding notations concerning q-lines are not given in explicit way.

Our reconstruction problem can be formulated as follows:

Consistency(p,q)
Instance: two directions p and q and two vectors $P = (p_{pmin}, \dots, p_{pmax}),\ Q = (q_{qmin}, \dots, q_{qmax})$.
Question: is there $F \in \mathcal{F}$ such that $rp(i) = p_i,\ rq(j) = q_j$ for any $i \in [pmin, pmax]$ and $j \in [qmin, qmax]$?

3 Properties of F and its subsets

Let δ and k be as in the previous section. We denote by $\mathcal{L}_l$ the subset of $\mathbb{Z}^2$ ($\mathcal{L}_l$ is called *p-q lattice*, see [2]) defined by:

$$\mathcal{L}_l = \{M \in \mathbb{Z}^2 \mid q(M) \equiv kp(M) \equiv l \pmod{\delta}\},$$

where $l \in \{0, \dots, \delta - 1\}$ (see Fig. 4). From this definition, $\mathbb{Z}^2$ is the union of δ disjoint p-q lattices and if $M \in \mathbb{Z}^2$ there is $l = l_M$ such that $M \in \mathcal{L}_{l_M}$. Obviously, if M and N belong to the same p-q-lattice $l_M = l_N$. Sometimes we also use l_i (l^j, resp.) to underline that $p = i$ ($q = j$, resp.) is a line of $\mathcal{L}_l$ with $l = l_i$ ($l = l^j$, resp.).

By considering the cumulated sums of the projections, we find conditions so that a point of $\mathbb{Z}^2$ belongs to $F \in \mathcal{F}$. Let $M = (i, j)$ be the point of Δ such that $p(M) = i,\ q(M) = j$. We denote the cumulated sums of the projections p_i and q_j as follows:

$$S_{0,l}(i) = \sum_{i' \le i} p_{i'},$$

$$S_{1,l}(j) = \sum_{j' \le j} q_{j'},$$

$$S_{2,l}(i) = \sum_{i' \ge i} p_{i'},$$

$$S_{3,l}(j) = \sum_{j' \ge j} q_{j'},$$

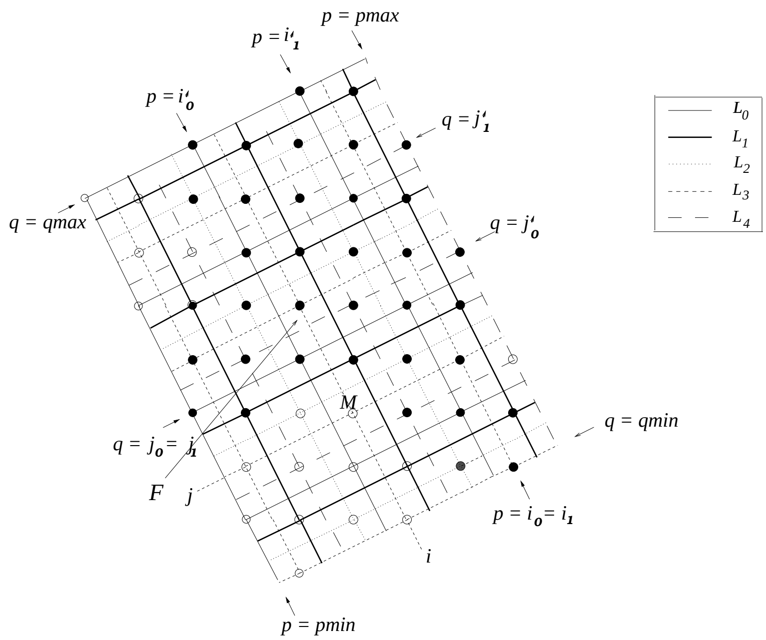

Figure 4: The integer lattice $\mathbb{Z}^2$ is the union of $\delta = 5$ disjoint p-q lattices, where $p = 2x+y$ and $q = -x+2y$.

where i' and j' in the sums are such that $l_{i'} = l$ and $l^{j'} = l$.

$$A_l = S_{0,l}(pmax_l) = S_{1,l}(qmax_l) = S_{2,l}(pmin_l) = S_{3,l}(qmin_l) \tag{3.1}$$

is the number of points of $F \cap \mathcal{L}_l$, and $pmax_l$ ($pmin_l$, resp.) is defined as $pmax$ ($pmin$, resp.) restricted to the indices i such that $l_i = l$ and similarly $qmax_l$ ($qmin_l$, resp.) is defined as $qmax$ ($qmin$, resp.) restricted to the indices j such that $l^j = l$.

Thus, if this condition on the cumulated sums is not verified there is no discrete set whose projection vectors are P and Q.

Lemma 3.1 *Let $M = (i,j) \in \Delta$. If $S_{t,l_M}(i) + S_{t+1,l_M}(j) > A_{l_M}$, then $F \cap Z_t(M) \neq \emptyset, t = 0,1,2,3$.*

Proof. If $F \cap Z_0(M) = \emptyset$ (see Fig. 4), then $S_{0,l_M}(i) + S_{1,l_M}(j) = |F \cap \mathcal{L}_{l_M} \cap (Z_3(M) \cup Z_1(M))| \leq A_{l_M}$. Analogously, cases $t = 1,2,3$ can be proved. □

Keeping in mind Definition 2.1, Lemma 3.1 gives conditions so that a point $M \in \mathbb{Z}^2$ belongs to $F \in \mathcal{F}$. If F is Q-convex around $\mathcal{D} = \{p,q\}$, the knowledge of the positions of its bases allows us to reduce the number of the conditions to be verified in order to decide whether a point of Δ belongs to F. First, we define some subsets of F and then we show their properties. We point out that such definitions depend on the position of the bases of F. Therefore, we will analyze in detail one case. For a different position of the bases, by proceeding analogously, similar results can be proved. Furthermore we assume: $j_0 \leq j'_1$ (as in Fig. 4). No assumption are made concerning the q-bases.

The sets G and H

For each point $M = (i, j)$ such that $j_0 \leq j \leq j'_1$, we get that $Z_0(M) \cap PMIN \neq \emptyset$ and $Z_2(M) \cap PMAX \neq \emptyset$. From now on, it is enough to show that $Z_1(M) \cap F \neq \emptyset$ and $Z_3(M) \cap F \neq \emptyset$, in order to prove that M belongs to F. In the following part, we deal with lines containing at least a point, because this is the interesting case. From condition (3.1), for each line $p = i$ such that $p_i > 0$ we can define two q-indices, as follows:

$$a_i = \min\{j : l_i = l^j = l \text{ and } S_{1,l}(j) + S_{2,l}(i) > A_l\} \tag{3.2}$$

$$b_i = \max\{j : l_i = l^j = l \text{ and } S_{3,l}(j) + S_{0,l}(i) > A_l\}. \tag{3.3}$$

Lemma 3.2 *If $p_i > 0$, then $a_i \leq b_i$, for $i \in [pmin, pmax]$.*

Proof. By (3.2) we have that $S_{1,l}(a_i - \delta) + S_{2,l}(i) \leq A_l$. Since $S_{1,l}(a_i - \delta) = A_l - S_{3,l}(a_i)$ and $S_{2,l}(i) = A_l - S_{0,l}(i - \delta)$, the inequality can be rewritten as $S_{3,l}(a_i) + S_{0,l}(i - \delta) \geq A_l$. If $p_i > 0$, then $S_{0,l}(i - \delta) < S_{0,l}(i)$ and therefore, $S_{3,l}(a_i) + S_{0,l}(i) > A_l$. On the base of (3.3) this implies $a_i \leq b_i$. □

At this point, we consider separately the following two cases for the lines $p = i$:

- $j_0 \leq a_i$ and $j'_1 \geq b_i$;
- $a_i < j_0$ or $j'_1 < b_i$.

First we take the lines $p = i$ verifying $j_0 \leq a_i$ and $j'_1 \geq b_i$ into consideration; let

$$G = \{M = (i, j) : \ j_0 \leq a_i \leq j \leq b_i \leq j'_1\}.$$

If $a_i \leq j \leq b_i$, then $S_{3,l_M}(j) + S_{0,l_M}(i) > A_{l_M}$ and $S_{1,l_M}(j) + S_{2,l_M}(i) > A_{l_M}$. From Lemma 3.1 it follows that $Z_1(M) \cap F \neq \emptyset$ and $Z_3(M) \cap F \neq \emptyset$. Since $j_0 \leq j \leq j'_1$, $Z_0(M) \cap PMIN \neq \emptyset$ and $Z_2(M) \cap PMAX \neq \emptyset$ as we pointed at the beginning of this section; by definition 2.1 the point M belongs to F. So, G is a subset of F. (Notice that this set only depends on the p-bases and the projections of F.)

Lemma 3.3 *If the integer i verifies $j_0 \leq a_i \leq b_i \leq j'_1$, then $p_i > 0$ implies that there is at least a point M of G belonging to the line $p = i$.*

Now we consider the case $a_i < j_0$ or $j'_1 < b_i$. In this case we are not able to find some point of F but we are able to determine some points of Δ which surely do not belong to F. For this reason, we introduce the sets H_1 and H_2. We define H_1 as follows:

$$H_1 = \{M = (i, j) : \ a_i < j_0, \ j \leq a'_i - \delta p_i \text{ or } j \geq b'_i + \delta p_i\},$$

where a'_i and b'_i are defined by:

$$a'_i = \max\{j : \ l^j = l_i \text{ and } j \leq j_0\}$$

$$b_i' = \min\{j : \ l^j = l_i \text{ and } j \geq j_0\}$$

Moreover, we define H_2 by

$$H_2 = \{M = (i, j) : \ j_1' < b_i, \ j \leq a_i'' - \delta p_i \text{ or } j \geq b_i'' + \delta p_i\},$$

where:

$$a_i'' = \max\{j : \ l^j = l_i \text{ and } j \leq j_1'\}$$
$$b_i'' = \min\{j : \ l^j = l_i \text{ and } j \geq j_1'\}$$

Lemma 3.4 *The sets H_1 and H_2 are disjointed to F.*

Proof. Let $p = i$ be a p-line such that $p_i > 0$ and suppose $a_i < j_0$. Let M and N be such that $p(M) = p(N) = i$ and $q(M) = a_i'$ and $q(N) = b_i'$ (see Fig. 5). We have to

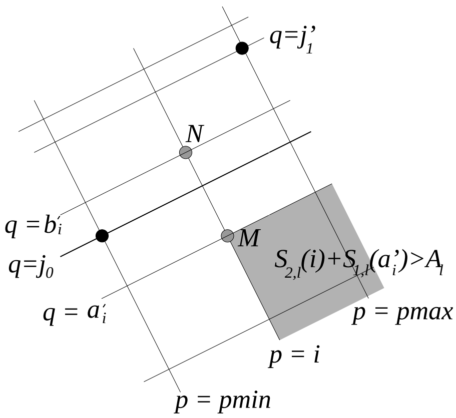

Figure 5: The a_i' and b_i' indices.

prove that $M \in F$ or $N \in F$. By $a_i < j_0$ and $l^{a_i} = l_i$ it follows that $a_i \leq a_i'$. Therefore, $S_{2,l}(i) + S_{1,l}(a_i') > A_l$ where $l = l_i$ and so $Z_1(M) \cap F \neq \emptyset$. Because $p_i > 0$, there is a point M' of F in the line $p = i$. If we suppose $q(M') \leq a_i'$ then we deduce $M \in F$. Conversely, if $q(M') \geq b_i'$ then $N \in F$. Finally, because there are p_i points on the line $p = i$, we deduce that the points of H_1 are not in F. In case $j_1' < b_i$ we can similarly show that $H_2 \cap F = \emptyset$. □

We denote the union of H_1 and H_2 by H.

4 Reconstruction algorithm

Now we describe the main steps of the reconstruction procedure. The first step is to check whether the given projections satisfy the condition (3.1) on the cumulated sums. Then, the algorithm chooses the p-bases or the q-bases depending on the sizes of the projection vector. (If $m < n$, the p-bases are chosen). The cost of this choice is $\min\{m^2, n^2\}$, number of possible positions of the bases. Furthermore we assume the p-bases are chosen.

At this point, we compute the sets G and H, since they only depend on the projections and the p-bases. This is made in $O(mn)$ time. We recall that G contains points of F and H contains points which surely do not belong to F. The algorithm builds two sets α and β such that $\alpha \subset F \subset \beta$ for each solution F. At the beginning $\alpha = G$ and $\beta = \Delta \setminus H$. Thus, α is made up of points of F, whereas $\beta - \alpha$ contains *indeterminate points* in the sense that we do not know whether they are or not in F. The idea of the algorithm consists in expanding α and reducing β by means of some operations. These operations take advantage of both the convexity constraint and vectors P, Q. We are going to define so called *filling operations*.

Let us denote the set of points of the intersection between $p = i$ ($q = j$) and β by β_i (β^j). We also denote the set of points of the intersection between $p = i$ ($q = j$) and α by α_i (α^j).

The following two operations impose the Q-convexity around $\mathcal{D}$ on the sets α and β and therefore, the simply convexity in the directions of $\mathcal{D}$. In fact, we remark that $\oplus'$ and $\ominus'$ restricted to a line work as $\oplus$ and $\ominus$ defined in [2].

- the operation $\oplus'$

  If $M \in \beta$ and $\forall t,\ \alpha \cap Z_t(M) \neq \emptyset$,

  then $\oplus'\alpha = \alpha \cup \{M\}$.

- the operation $\ominus'$

  If $M \notin \beta$ and $\exists t$ such that $\alpha \cap Z_{t-1}(M) \neq \emptyset,\ \alpha \cap Z_t(M) \neq \emptyset$ and $\alpha \cap Z_{t+1}(M) \neq \emptyset$,

  then $\ominus'\beta = \beta - Z_{t+2}(M)$.

The following operations are the "coherence" operations defined in [2]. They are executed for each line $p = i$ and for each line $q = j$. For example we fix a line $p = i$.

Let $M,\ N \in \alpha_i$ be such that $q(M) = \min\{j : (i,j) \in \alpha,\ j \in [qmin, qmax]\}$ and $q(N) = \max\{j : (i,j) \in \alpha,\ j \in [qmin, qmax]\}$;

let $M',\ N' \in \beta_i$ be such that $q(M') = \min\{j : (i,j) \in \beta,\ j \in [qmin, qmax]\}$ and $q(N') = \max\{j : (i,j) \in \beta,\ j \in [qmin, qmax]\}$;

- the operation $\otimes$ on $p = i$

  $\otimes\alpha = \alpha \cup \{(i,j) :\ q(N') - \delta(p_i - 1) \leq j \leq q(M') + \delta(p_i - 1)\}$.

- the operation $\odot$ on $p = i$

  if $\alpha_i = \emptyset$, then $\odot\beta = \beta$;

if $\alpha_i \neq \emptyset$, then $\odot\beta = \beta - \{(i,j):\ j \leq q(N) - \delta p_i \text{ or } j \geq q(M) + \delta p_i\}$.

The following operation allows us to delete in β a sequence of consecutive indeterminate points of $p = i$, when the sequence is shorter than the projection p_i.

- the operation $\odot'$ on $p = i$.
  If there exist $M, N \in \beta_i$ such that $q(N) - q(M) < \delta(p_i - 1)$ and $(i, q(M) - \delta)$ and $(i, q(N) + \delta)$ are not in β, then $\odot'\beta = \beta - \{(i,j):\ q(M) \leq j \leq q(N)\}$.
  In the other cases $\odot'\beta = \beta$.

The filling operations on the q-lines are defined analogously.

The algorithm performs these operations on the p-lines and on the q-lines in the following order: $\oplus'$, $\ominus'$, $\otimes$, $\odot$, $\odot'$. The application is repeated iteratively until $\alpha \not\subset \beta$ or they produce no further changes in α and β. The computational cost of the procedure for performing the filling operations is $O((mn)^2)$. If we obtain $\alpha \not\subset \beta$, then there is no discrete set of $\mathcal{F}$ containing the chosen p-bases and having projections P, Q. Therefore, the algorithm chooses a different position of the p-bases and tries again.
If $\alpha = \beta$ is a Q-convex discrete set around $\mathcal{D}$, then $\alpha = F$ and so there is at least one solution of the problem (the algorithm reconstructs one of them). Finally, we can obtain the case in which α and β are invariant with respect to the filling operations and $\alpha \subset \beta$, so that $\beta - \alpha$ is not empty.

4.1 The case: $\alpha \subset \beta$

In the following table are shown four types of lines; black, gray and white-colored points represent a point of α, an indeterminate point and a point which does not belong to β, respectively. More precisely, the line $p = i$ is of type:

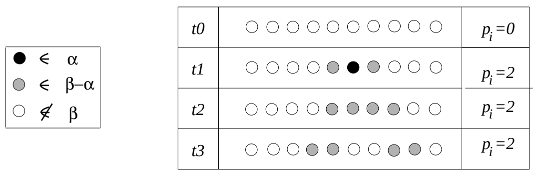

Table 1: The several types of lines.

- $t0$, if $p_i = 0$.
- $t1$, if $\alpha_i \neq \emptyset$ and $(\beta_i - \alpha_i) \neq \emptyset$ is made up of two separated sequences each of them containing $p_i - k$ points, where k is the cardinality of α_i.
- $t2$, if $\alpha_i = \emptyset$ and β_i is connected and contains exactly $2p_i$ points.
- $t3$, if $\alpha_i = \emptyset$ and β_i consists of two separated sequences of p_i consecutive points.

If we regard Lemma 3.3 and 3.4 keeping in mind Table 1, we can claim that:

- after performing the filling operations, if $j_0 \le a_i \le b_i \le j_1'$, then $p_i > 0$ implies that the line of equation $p = i$ is of type $t1$;
- if $a_i < j_0$ or $b_i > j_1'$, by the definition of H, $p_i > 0$ implies that the line of equation $p = i$ is of type $t1$ or $t2$.

We summarize the obtained results in the following proposition.

Proposition 4.1 *After performing the filling operations, each line having equation $p = i$ is of type $t0$ or $t1$ or $t2$, $i \in [pmin, pmax]$.*

Let $p = i$ be any p-line containing indeterminate points. (It exists because $\beta - \alpha \neq \emptyset$.) From Proposition 4.1, we deduce that:

$$|\beta_i| = 2p_i - |\alpha_i| \text{ for all } i \in [pmin, pmax].$$

We remark that $\alpha_i = 0$ when $p = i$ is of type $t2$ or $t3$. By summing on i we have

$$|\beta| = 2A - |\alpha|, \tag{4.4}$$

where $A = \sum p_i$. Consider now the q-lines and let $q = j$ be the equation of any line containing indeterminate points. Thanks to the operations $\otimes$ and $\odot'$, we have :

$$|\beta^j| \ge 2q_j - |\alpha^j|$$

and therefore,

$$|\beta| = \sum_j |\beta^j| \ge \sum_j (2q_j - |\alpha^j|) = 2A - |\alpha|.$$

By (4.4) we deduce:

$$|\beta^j| = 2q_j - |\alpha^j| \text{ for all } j \in [qmin, qmax],$$

otherwise we get a contradiction. We note that this result allows us to establish the type of the q-lines. In fact, when $|\alpha^j| > 0$ we know that $q = j$ is a line of type $t1$. If $|\alpha^j| = 0$ then we have $|\beta^j| = 2q_j$ so that by means of the operation $\odot'$ the set β^j is made up of two sequences, $seq1$ and $seq2$, having the same length, being either consecutive (in this case $q = j$ of type $t2$) or separate (in this case $q = j$ of type $t3$); see Fig. 6.

Proposition 4.2 *After performing the filling operations, each line having equation $q = j$ is of type $t0$ or $t1$ or $t2$ or $t3$, $j \in [qmin, qmax]$.*

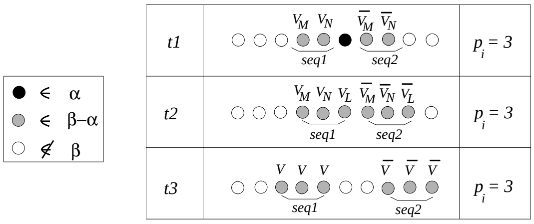

Figure 6: Relationship between indeterminate points of $seq1$ and $seq2$.

4.2 Expression of the Q-convexity by a 2-SAT formula

The algorithm has been halted and $\alpha \subset \beta$. In this case we are not yet able to say whether there is a solution, because $\beta - \alpha$ is not empty. Then, determining the existence of a solution of our reconstruction problem is linked to determining the existence of an evaluation v of $\beta - \alpha$ for which the new α belongs to $\mathcal{F}$ and has projection vectors P and Q. We denote the set α obtained by means of v by $v(\alpha)$. Consider any line containing indeterminate points. What we do is to associate literals to the indeterminate points of $\beta - \alpha$ in such a way a variable or its negation expresses exactly the state : $M \in v(\alpha)$. If the indeterminate points into consideration belong to a line of type $t1$ or $t2$, we proceed as shown in [2]: if V is the literal associated to M, then the point $(i, q(M) + \delta p_i)$ is associated to the variable $\overline{V}$ (the negation of V). In case of indeterminate points belonging to a line of type $t3$, the same literal is associated to all the points belonging to the same sequence (see Fig. 6).

Each evaluation of the boolean variables gives a solution satisfying the projections, but not satisfying the convexity constraints. We recall that F is Q-convex around p and q if $M \notin F$ implies that there is t such that $Z_t(M) \cap F = \emptyset$. So, if M is an indeterminate point or if $M \notin \beta$, it is necessary to impose the Q-convexity constraints for M. This is made by constructing a boolean formula. Considering the several types of lines, the following four cases can arise (we denote the literal associated to M by V_M):

Case t1,2/t1,2: $M = (i, j)$, $p(M) = i$ is of type $t1$ or $t2$ and $q(M) = j$ is of type $t1$ or $t2$. If the line $p = i$ is of type $t1$ then this line contains a point of α. If the line is of type $t2$, then there are two consecutive indeterminate points (the $(p_i/2)$-th point and $(p_i/2 + 1)$-th point of the sequence) such that at least one of them is in F. So, in the two cases one of the two semi-lines $(p = i, q \leq j)$ or $(p = i, q \geq j)$ contains a point of F. We suppose, for example that $(p = i, q \geq j)$ contains a point of F. In the same way we can suppose that $(p \geq i, q = j)$ contains a point of F (see Fig. 7).

- Let $M \in \beta - \alpha$ (see Fig. 7a)). If $\alpha \cap Z_0(M) \neq \emptyset$, then we impose the clause $V_M = 1$, else for each indeterminate point N of $Z_0(M)$ we impose the clause $V_N \rightarrow V_M$.
- Let $M \notin \beta$ (see Fig. 7b)) . If $\alpha \cap Z_0(M) \neq \emptyset$, then the formula is FALSE (there is no solution for the considered choice of the positions of the p-bases), else for each

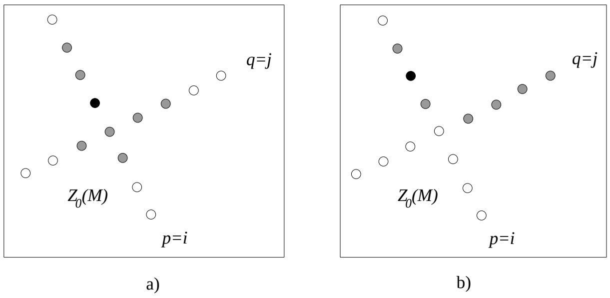

Figure 7: $p(M) = i$ is of type $t1$ and $q(M) = j$ is of type $t2$. a) $M \in \beta - \alpha$. b) $M \notin \beta$.

indeterminate point N of $Z_0(M)$ we impose the clause $V_N = 0$.

Case t1,2/t0: $M = (i, j)$, $p(M) = i$ is of type $t1$ or $t2$ and $q(M) = j$ is of type $t0$. In this case $M \notin \beta$; suppose, for instance, $p(M) = i$ and $q(M) = j$ as in Fig. 8. For each $N' \in Z_0(M) \cap \beta$ and $N \in Z_1(M) \cap \beta$ we impose the clauses: $V_N \rightarrow \overline{V_{N'}}$

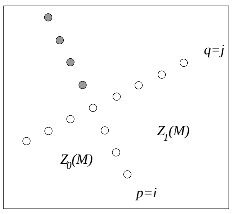

Figure 8: $M \notin \beta$, $p(M) = i$ is of type $t2$ and $q(M) = j$ is of type $t0$.

and $V_{N'} \rightarrow \overline{V_N}$.

Case t1,2/t3: $M = (i, j)$, $p(M) = i$ is of type $t1$ or $t2$ and $q(M) = j$ is of type $t3$.

- Let $M \in \beta - \alpha$. This case is similar to the case $t1, 2/t1, 2$, because we know that only one among the four zones could contain no point of the F. For instance, in Fig. 9a) $Z_0(M)$ is the one in question.

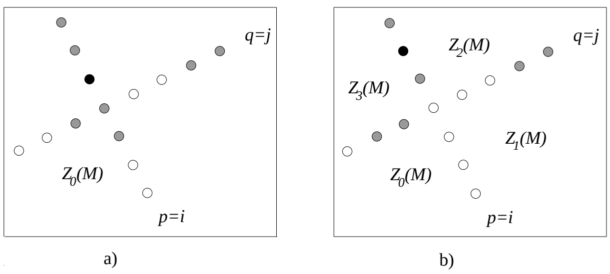

Figure 9: $p(M) = i$ is of type $t1$ and $q(M) = j$ is of type $t3$. a) $M \in \beta - \alpha$. b) $M \notin \beta$.

- Let $M \notin \beta$. Suppose, for instance, case in Fig. 9b) arises. Let V be the unique literal associated to each indeterminate point of $Z_0(M) \cap Z_3(M)$; thus, $\overline{V}$ is associated to each indeterminate point of $Z_1(M) \cap Z_2(M)$. For each $N' \in Z_0(M) \cap \beta$ and $N \in Z_1(M) \cap \beta$ we impose the clauses : $V \to \overline{V_{N'}}$ and $\overline{V} \to \overline{V_N}$.

Since $PMAX$ and $PMIN$ have been chosen, for each $M \in \Delta$ there are at least two zones among the four zones $Z_0(M), Z_1(M), Z_2(M), Z_3(M)$ having not empty intersection with α. We use this property for the last case.

Case t0/t0,3: $M = (i, j)$, $p(M) = i$ is of type $t0$ and $q(M) = j$ is of type $t0$ or $t3$.

In this case $M \notin \beta$. Fig. 10 shows the case wherein $p(M) = i$ is of type $t0$ and $q(M) = j$ is of type $t3$ and $j < j_0$.

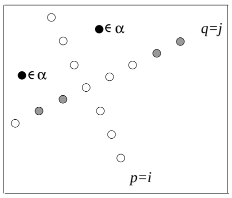

Figure 10: $M \notin \beta$, $p(M) = i$ is of type $t0$ and $q(M) = j$ is of type $t3$.

a. If the four zones have not empty intersection with α, then the formula is FALSE (there is no solution for the considered choice of the positions of the p-bases).

b. If only one zone $Z_t(M)$ has empty intersection with α, then for each N of $Z_t(M)$ we impose the clause $V_N = 0$.

c. The case where two zones $Z_t(M)$, $Z_s(M)$ have empty intersection with α is similar to the case $t1/t0$. Then, for each $N \in \beta \cap Z_t(M)$ and $N' \in \beta \cap Z_s(M)$ we impose the clauses $V_N \to \overline{V_{N'}}$, $V_{N'} \to \overline{V_N}$.

The algorithm builds a boolean formula which is a collection of clauses, each of them expressing a Q-convexity constraint for a point M. Each assignment of values of the variables satisfying the formula gives rise to a solution F of our reconstruction problem. In fact, F has projections P and Q and it is easy to check that $F \in \mathcal{F}$. We have just shown that we are able to express the Q-convexity constraints by means of a 2SAT formula; now we analyze the computational complexity for constructing the formula and solving the satisfiability problem. Since we impose the Q-convexity constraints for each point $M \in \beta - \alpha$ or $M \notin \alpha$, the number of variables is less than or equal to $mn - |\alpha|$. The clauses deriving from cases $t1,2/t1,2$, $t1,2/t3$, a, b of cases $t0/t0,3$ can be found in $O((mn)^2))$.

In all the other cases ($t1,2/t0$, c of $t0/t0,3$), for each p-line (or q-line) of type $t0$ we build clauses for the points N, N' in $O((mn)^2)$ time. Since there are at most $m+n$ lines of type $t0$, these clauses take $O((mn)^2(m+n))$ time to be constructed. Moreover, 2SAT can be solved in linear time with respect to the number of variables and clauses and so the satisfiability of the formula can be checked in $O((m+n)(mn)^2)$. Finally, it takes $O((mn)^2(m+n))$ time to know if the chosen p-base-position gives a solution or not. So we have :

Proposition 4.3 *The problem of reconstructing a discrete set which is Q-convex around $\{p,q\}$ is solvable in $O(min\{m^2,n^2\}(mn)^2(m+n))$ time.*

We point out that if F is indivisible the complexity of the algorithm is $O(\min\{m^2,n^2\}$ $(mn)^2)$, because cases $t1,2/t1,2,3$ are the only possible ones.

5 More than two projections

Let $\mathcal{D}$ be a set of three rational directions p,q and r, whose the last one is defined by $r(M)=ex_M+fy_M$, with $e,f\in\mathbb{Z}$, $\gcd(e,f)=1$. We still assume $\det(p,r)=af-be\neq 0$ and $\det(q,r)=cf-de\neq 0$. Now a point M of $\mathbb{Z}^2$ is intersection of three lines having equations $p(M)=i$, $q(M)=j$ and $r(M)=k$. Consider the pair (p,q) of directions: the p-q lattices ${\mathcal{L}_l}^{(p,q)}$ are defined and a point M selects the four zones ${Z_t}^{(p,q)}(M)$, with $t=0,1,2,3$. Analogously, we define q-r lattices ${\mathcal{L}_l}^{(q,r)}$ and r-p lattices ${\mathcal{L}_l}^{(r,p)}$, by considering the pairs (q,r) and (r,p), respectively. Moreover, for these lattices we also have that M selects the zones ${Z_t}^{(q,r)}(M)$ and ${Z_t}^{(r,p)}(M)$, with $t=0,1,2,3$.

Definition 5.1 *F is Q-convex around $\{p,q,r\}$ if it is Q-convex around $\{p,q\}$, $\{q,r\}$ and $\{r,p\}$.*

Our algorithm can be easily extended in order to work in case of a set $\mathcal{D}$ of three or more directions for reconstructing discrete sets which are Q-convex around $\mathcal{D}$. We remark that, if we choose the positions of the p-bases with respect to ${\mathcal{L}_l}^{(p,q)}$, by definition of p-bases, their positions are also automatically fixed with respect to ${\mathcal{L}_l}^{(r,p)}$ (see Fig. 11), that is, the

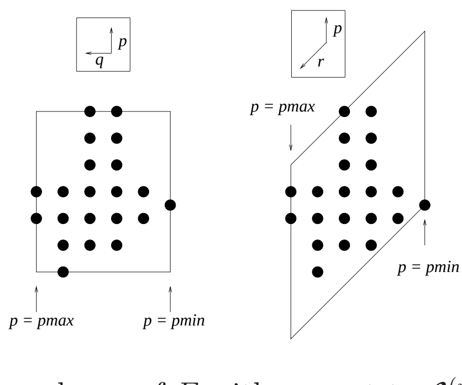

Figure 11: The p-bases of F with respect to $\mathcal{L}^{(p,q)}$ and $\mathcal{L}^{(r,p)}$.

choice does not depend on the lattices we are considering. Thus, the algorithm chooses the positions of the p-bases and then it determines the sets $G^{(p,q)}$, $H^{(p,q)}$, $G^{(r,p)}$ and $H^{(r,p)}$. It sets $\alpha=G^{(p,q)}\cup G^{(r,p)}$ and $\beta=\Delta-\{H^{(p,q)}\cup H^{(r,p)}\}$; after that, the filling operations are performed in ${\mathcal{L}_l}^{(p,q)}$, then in ${\mathcal{L}_l}^{(r,p)}$ and in ${\mathcal{L}_l}^{(q,r)}$ and the application is repeated until no more change are possible. As above, by considering ${\mathcal{L}_l}^{(p,q)}$ and ${\mathcal{L}_l}^{(r,p)}$, we have that p-lines, q-lines and r-lines are of types $t0$ or $t1$ or $t2$ or $t3$. By Definition 5.1, the Q-convexity constraints can be imposed by a boolean formula whose clauses express the Q-convexity around $\{p, q\}$, $\{r, q\}$ and $\{q, r\}$. We showed that in case (p, q), and therefore in case (r, p), we built a 2SAT formula. For the clauses expressing the Q-convexity around $\{q, r\}$ we have to analyze the cases **t0,3/t0,3**, because the other ones are the same as in case $|\mathcal{D}| = 2$.

Case t0,3/t0,3: $M = (j, k)$, $r(M) = k$ and $q(M) = j$ are of type $t0$ or $t3$.

- First, suppose that $M \in \beta$, in this case the lines $r(M) = k$ and $q(M) = j$ are of type $t3$ and we proceed as in case $t1,2/t1,2$ of section 4.2. In fact, in Fig. 12 only $Z_0(M)$ could contain no point of F.

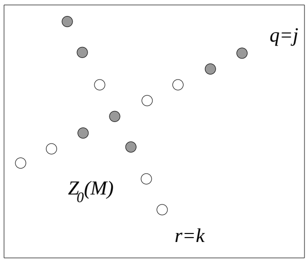

Figure 12: $r(M) = k$ is of type $t3$, $q(M) = j$ is of type $t3$ and $M \in \beta - \alpha$

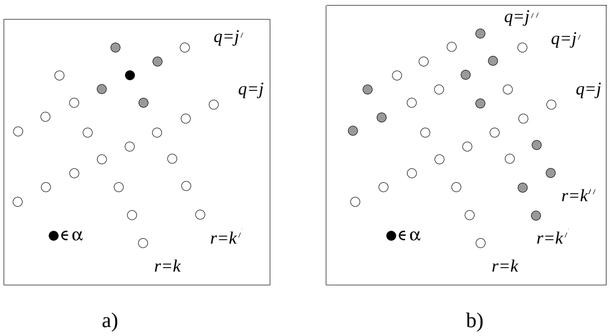

Figure 13: $M \notin \beta$, $r(M) = k$ and $q(M) = j$ are of type $t0$.

- Now we suppose $M \notin \beta$. The p-bases are chosen so that there is at least one among the zones ${Z_0}^{(q,r)}(M)$, ${Z_1}^{(q,r)}(M)$, ${Z_2}^{(q,r)}(M)$ and ${Z_3}^{(q,r)}(M)$, which contains a point of α. Suppose ${Z_0}^{(q,r)}(M) \cap \alpha \neq \emptyset$, as in Fig. 13a) and b) and consider the lines $q = j'$ and $r = k'$ with $j' \geq j$ and $k' \geq k$.
  - If one of these lines is of type $t1$ or $t2$ (see Fig. 13a)), or $t3$ and more than half of its indeterminate points belong to the same zone among the ones defined by $q = j$ and $r = k$, then we can find a $t \neq 0$ such that ${Z_t}^{(q,r)}(M) \cap F \neq \emptyset$ for any solution F. So, this is similar to case $t0/t0,3$ of section 4.2.
  - Otherwise, if case in Fig. 13b) arises, the literals which appear in ${Z_2}^{(q,r)}(M)$ are exactly the negations of those which appear in ${Z_1}^{(q,r)}(M)$ and ${Z_3}^{(q,r)}(M)$. By imposing the Q-convexity, one of this zones must be empty. Therefore, for each indeterminate point N' and N'' in ${Z_2}^{(q,r)}(M)$, we can express the Q-convexity by means of the clauses: $V_{N'} \leftrightarrow V_{N''}$.

The complexity of this algorithm is $O(\min\{m^2, n^2\}(mn)(mn+mo+no+(mn)(m+n+o)))$ with $o = rmax - rmin + 1$. We can generalize the algorithm in order to work with any number d of directions. The question is: taken d directions $u_1, \ldots, u_d$ and d vectors $U_1 \in \mathbb{N}^{n_1}, \ldots, U_d \in \mathbb{N}^{n_d}$, is there a set $F \subset \mathcal{F}$ whose projections in the d directions are the given vectors? Our algorithm solves the problem constructing a solution in

$$O\left(n_i^2(n_i n_j)\left(\sum_{k\neq k'} n_k n_{k'} + d n_i n_j \sum_k n_k\right)\right)$$

where i is such that $n_i = \min_k(n_k)$ and $n_j = \min_{j\neq i}(n_k)$. More simply if $n = \max(n_k)$ then the complexity is less than $O(n^7 d^2)$. In fact the power index of d is 2 because we have to check the Q-convexity for all the couples (u_i, u_j), $\forall i, j$.
The following lemma shows that when the set F is indivisible in one direction it is not necessary to look for all the couples.

Lemma 5.2 *Let F be p-indivisible. F is Q-convex around $\{p, q, r\}$ if it is Q-convex around $\{p, q\}$ and $\{p, r\}$.*

Proof. Let M be such that $p(M) = i$, $q(M) = j$ and $r(M) = k$ and $M \notin F$. We can suppose that $Z_0^{(p,r)}(M) = Z_0^{(p,q)}(M) \cup Z_0^{(q,r)}(M)$ because it is true up-to a translation of the indices. So we are in the configuration of Fig. 14.

If F is Q-convex around $\{p, q\}$, then there exists t such that ${Z_t}^{(p,q)}(M) \cap F = \emptyset$. Suppose ${Z_0}^{(p,q)}(M) \cap F = \emptyset$, as shown in Fig. 14. (The case ${Z_2}^{(p,q)}(M) \cap F = \emptyset$ is

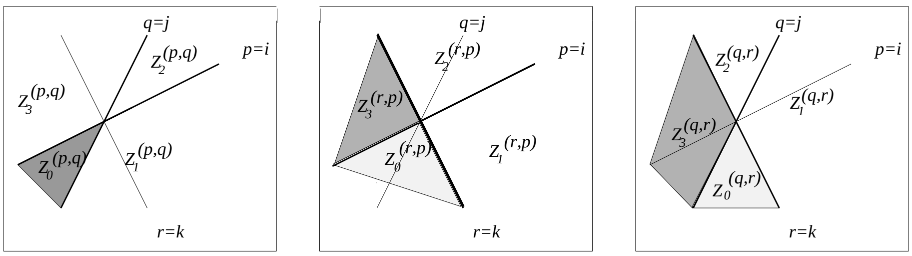

Figure 14: The zones around $\{p, q\}$, $\{r, p\}$, $\{q, r\}$.

symmetric to the previous one, while cases ${Z_1}^{(p,q)}(M) \cap F = \emptyset$ and ${Z_3}^{(p,q)}(M) \cap F = \emptyset$ are banal because these quadrants contain quadrants for the couple (q, r).) Now take directions $\{p, r\}$ into consideration. By the hypothesis, there is t such that ${Z_t}^{(r,p)}(M) \cap F = \emptyset$. Suppose ${Z_0}^{(r,p)}(M) \cap F = \emptyset$. Then, ${Z_0}^{(q,r)}(M) \cap F = \emptyset$ because of $Z_0^{(r,p)}(M) = Z_0^{(p,q)}(M) \cup Z_0^{(q,r)}(M)$ and so F is Q-convex around $\{p, q, r\}$. Similarly, if we suppose that ${Z_3}^{(r,p)}(M) \cap F = \emptyset$ we have ${Z_3}^{(q,r)}(M) = {Z_3}^{(r,p)}(M) \cup Z_0^{(p,q)}(M)$ and so ${Z_3}^{(q,r)}(M) \cap F = \emptyset$. Finally, if ${Z_1}^{(r,p)}(M) \cap F = \emptyset$ or ${Z_2}^{(r,p)}(M) \cap F = \emptyset$, we also get a contradiction since we deduce that $p_i = 0$. □

So, when the projections along p have no zero values it is enough to work considering ${\mathcal{L}_l}^{(p,q)}$ and ${\mathcal{L}_l}^{(r,p)}$.

In case of d directions, one can easily show similarly to Lemma 5.2, by induction, that when F is u_i-indivisible, F is Q-convex around $\mathcal{D}$ if it is Q-convex around $\{u_i, u_j\}$ $\forall j \neq i$. Thus, it gives an algorithm of complexity :

$$O\left(n_i^3(n_i n_j)\left(\sum_k n_k + dn_j\right)\right)$$

where $n_j = \min_{k \neq i} n_k$. If $n = \max(n_k)$ the complexity is $O(n^6 d)$ which is much better than in the general case.

6 Conclusion and related problems

In [9] G. J. Woeginger has proved that the reconstruction problem by directions $(1,0)$ and $(0,1)$ on the class of discrete sets horizontally and vertically convex (denoted by (h,v)) is NP-complete. Conversely, we have shown that the same problem on the class of Q-convex discrete sets is solvable in polynomial time. A question derives by observing that, among the sets which are convex, (h,v) is the more general one. Then, is there a class which is more general than $\mathcal{F}$ and on which the reconstruction problem is still solvable in polynomial time?

A new result of Daurat [6] states when subsets of $\mathcal{F}$ are uniquely determined by the data. In this paper we establish the difficulty of the related algorithmic problem, showing that the reconstruction problem in $\mathcal{F}$ is solvable in polynomial time. Thus, the most important application of the proposed algorithm is that it allows to reconstruct a convex discrete set $F = convF \cap \mathbb{Z}^2$ from its projections taken in any certain set of directions, so answering the question proposed by Gritzmann during the workshop held in Dagsthul in 1997. In fact, as a consequence of the uniqueness result, one can reconstruct a Q-convex set and then check whether it is also convex; these steps are made in polynomial time. Therefore, the problem of reconstructing a convex discrete set from its projections is solvable in polynomial time when the projections are taken in any set of seven directions, or certain sets of four directions.

References

[1] E. Barcucci, S. Brunetti, A. Del Lungo, and M. Nivat, " Reconstruction of Discrete Sets from Three or More X-rays", *Proc. of CIAC 2000*, LNCS 1767 (2000) 199-210.

[2] E. Barcucci, A. Del Lungo, M. Nivat and R. Pinzani," Reconstructing convex polyominoes from their horizontal and vertical projections", *Theor. Comp. Sci.*, **155**, (1996) 321-347.

[3] E. Barcucci, A. Del Lungo, M. Nivat, R. Pinzani, "Medians of polyominoes: A property for the reconstruction", *Int. J. Imaging Systems and Techn.*, **8**, (1998) 69-77.

[4] M. Chrobak, C. Dürr, "Reconstructing hv-Convex Polyominoes from Orthogonal Projections", *Inf. Proc. Lett.* **(69) 6**, (1999) 283-289.

[5] A. Daurat, "The Q-convexity", preprint LLAIC1, (1999).

[6] A. Daurat, "Uniqueness of the reconstruction of Q-convexes from their projections", preprint LLAIC1, (2000).

[7] A. Daurat, A. Del Lungo, M. Nivat, "Medians of discrete sets according to a linear distance", to appear in *Disc. and Comp. Geom.* (1999).

[8] R. J. Gardner, P. Gritzmann and D. Prangenberg, "On the computational complexity of reconstructing lattice sets from their X-rays," *Disc. Math.* **202**, (1999) 45-71.

[9] G.J. Woeginger, "The reconstruction of polyominoes from their orthogonal projections," *Technical Report SFB-65, TU Graz, Graz* (1996)